J. Fluid Mech. (1972), *vol.* 54, *part* 1, *pp.* 81–91
Printed in Great Britain

Some numerical solutions of a variable-coefficient Korteweg–de Vries equation (with applications to solitary wave development on a shelf)

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(Received 29 July 1971 and in revised form 7 March 1972)

Some numerical solutions of a variable-coefficient Korteweg–de Vries equation are presented. This particular equation was derived by the author recently (Johnson 1972) in an attempt to describe the development of a single solitary wave moving onto a shelf. Soliton production on the shelf was predicted and this is confirmed here. Results for two and three solitons are reproduced and two intermediate shelf depths are also considered. In these latter two cases both solitons and an oscillatory wave occur. One of the profiles corresponds to the integrations performed by Madsen & Mei (1969) and a comparison is made.

1. Introduction

In the last few years, considerable interest has centred on the initial-value problem for the Korteweg–de Vries (KV) equation:

$$u_t + uu_x + u_{xxx} = 0, \quad u(x,0) = f(x). \tag{1}$$

The greatest success has been achieved by considering the solution $u(x,t)$ such that

$$u(x,t), f(x) \to 0 \quad \text{as} \quad x \to \pm\infty.$$

In particular, the papers published by Miura, Gardener & Kruskal (1968, 1969, 1970) review most of the recent work on this problem. The 'soliton' formation from a suitable initial condition is now well known, but it is worth noting that it was first observed in numerical solutions of (1) (see Zabusky & Kruskal 1965; Zabusky & Galvin 1971).

Recently, Johnson (1972, henceforth referred to as I) considered the problem of a solitary wave moving onto a shelf and derived a Korteweg–de Vries equation with variable coefficients to describe this motion. This particular problem was first discussed in detail by Madsen & Mei (1969), who compared numerical results with some experimental data. Reasonable agreement was obtained. For the numerical solutions they integrated what were essentially the full inviscid equations.

However, by making use of formal asymptotic methods, a single equation can be derived (see I) to describe this phenomenon. This equation takes the form

$$u_X + d^{-\frac{7}{4}} uu_\xi + d^{\frac{1}{2}} u_{\xi\xi\xi} = 0, \tag{2}$$

where $d(X)$ is the local variable depth, and the observed wave amplitude is proportional to $ud^{-\frac{1}{4}}$. It can then be shown that, if a solitary wave moves over the uniform depth ($d = 1$) without changing shape before reaching the shelf, it breaks up into a finite number of solitons (n) on the shelf provided that

$$d_0 = [\tfrac{1}{2}n(n+1)]^{-\frac{4}{9}}, \tag{3}$$

where d_0 is the depth of the shelf and n is an integer ($n \geqslant 1$). This result has also been derived recently by Tappert & Zabusky (1971)†, who discuss two other problems concerning the Korteweg–de Vries equation and inhomogeneous media. In this paper they assume that the appropriate constant-coefficient KV equation is applicable in the second uniform region (a point discussed in some detail in I). From (3) it follows that the shelf must be shallower than the uniform depth (i.e. $d_0 < 1$). The number n of solitons formed is then independent of the shape of the shelf formation.

The purpose of this paper is to describe some numerical solutions of (2) with a solitary-wave initial condition,

$$u(\xi, 0) = 12\,\mathrm{sech}^2\,\xi, \tag{4}$$

and shelves of various depths. An attempt is made to confirm the soliton formation and, more importantly, to discuss the situation when the shelf depth (d_0) does not satisfy (3).

The numerical scheme used is of standard form, being centred in both the time-like co-ordinate X and the space-like co-ordinate ξ. This scheme was used by Zabusky & Kruskal (1965) for their studies on the constant-coefficient KV equation. Very recently, Vliegenthart (1971) has discussed various numerical procedures for the KV equation, in particular the one used by Zabusky & Kruskal. He has derived a stability criterion for the scheme which has been confirmed by the present author in some trial runs.

2. Basis for the equation

Briefly the genesis of the variable-coefficient KV equation (2) is as follows. Consider a small amplitude motion defined by the parameter ϵ (so that ϵ = maximum amplitude/depth) on a uniform depth of water. For shallow-water waves (δ = depth/wavelength $\ll 1$), it is possible to derive the classical Korteweg–de Vries equation in a far-field (distance or time $O(\epsilon^{-1})$, linearized characteristic $O(1)$) if we choose $\delta^2 = O(\epsilon)$. This *ad hoc* assumption (Korteweg & de Vries 1895) about the two (independent) parameters ensures that nonlinear (ϵ) and dispersive (δ^2) effects are of the same order. Changes of undisturbed depth do not occur.

If the same analysis is pursued but the depth is now allowed to vary slowly on the scale of ϵ, then the far-field (distance $O(\epsilon^{-1})$) approximation incorporates the effects of changing depth. The near-field first approximation is, of course, unaltered since the depth approaches a constant as $\epsilon \to 0$. The resulting

† The author is indebted to a referee for introducing him to this paper.

equation then has terms depending on the depth, nonlinearity and dispersion, all being of order unity, and can be written in terms of the independent variables

$$\left.\begin{aligned} &X = \epsilon x \quad \text{(far-field distance co-ordinate)}, \\ &\xi = \int_0^x d^{-\frac{1}{2}}(\epsilon x)\,dx - t = O(1) \quad \text{(appropriate characteristic co-ordinate)}, \end{aligned}\right\} \tag{5}$$

where x and t are the original (non-dimensional) space and time variables respectively. When the attenuation factor of $d^{-\frac{1}{4}}$ (usually called Green's Law) is removed the final equation takes the form

$$u_X + d^{-\frac{7}{4}} u u_\xi + d^{\frac{1}{2}} u_{\xi\xi\xi} = 0, \quad d = d(X), \tag{2a}$$

where $u(\xi, X)$ is proportional to the elevation of the wave. Note that in (2a) the region of changing depth ($d = O(1), X = O(1)$) is of the same order of magnitude as the 'period' ($\xi = O(1)$) of the wave. However, in the original non-dimensional (physical) variables the region of changing depth is extended (having length of $O(\epsilon^{-1}), \epsilon \to 0$) and the 'wavelength' is still $O(1)$. The change in depth need not be sudden even as a function of the far-field co-ordinate X. In fact it is straightforward to show that, in particular, the change in depth may occur asymptotically rapidly or slowly.

The derivation outlined above is explained in detail in I, as is the application of classical KV theory to (2a). We note that if the depth $d(X)$ is constant then (2a) is the standard KV equation. If d varies over some finite range of X and outside this range takes constant values, and the initial wave form is fixed, then solitons can appear on the shelf. However, this is only true if $d_0 < 1$, where

$$\left.\begin{aligned} d(X) &= 1, \quad X \leqslant 0, \\ d &= d(X), \quad 0 \leqslant X \leqslant X_0 \quad (X_0 < \infty), \\ d(X) &= d(X_0) = d_0, \quad X \geqslant X_0. \end{aligned}\right\} \tag{6}$$

In I it is shown that if the depth of the shelf is given by $d_0 = [\frac{1}{2}n(n+1)]^{-\frac{4}{9}}$, with n an integer, then exactly n solitons will develop and d_0 is then an 'eigendepth'. In fact the solution asymptotically far along the shelf ($X \to \infty$) is the same as if the initial wave were placed directly on the shelf itself and then allowed to develop. This equivalence to the flow on the shelf alone enables the solution for the solitons to be found from Miura's theory for the constant-coefficient KV equation. This is simply because the relevant equation is (2a) with d_0 replacing $d(X)$, and asymptotically the solution is independent of the form of $d(X)$ in $O < X < X_0$. If the initial wave profile is the solitary wave

$$u(\xi, 0) = 12\,\text{sech}^2\,\xi,$$

which is a steady-state solution of

$$u_X + u u_\xi + u_{\xi\xi\xi} = 0,$$

then the n solitons (predicted by (2a)) can easily be shown to have amplitudes

$$u_{\max} = \frac{24}{n(n+1)} m^2 \quad (m = 1, 2, \ldots, n). \tag{7a}$$

The result $(7a)$ indicates a rather interesting phenomenon. As more solitons are formed ($d_0 \to 0$) the maximum amplitude approaches a finite value and the minimum amplitude approaches zero: the maximum amplitude (given by $m = n \to \infty$) is 24 and the minimum amplitude ($m = 1, n \to \infty$) is 0 (always remembering that the limit $d_0 \to 0$ might eventually invalidate the KV equation, and that the experimentally observed amplitudes are proportional to $u_{\max} d_0^{-\frac{1}{4}}$, which is clearly non-uniform).

Provided that only solitons are formed, that is, if there is an integer solution of (3) for given d_0, the picture seems clear and we have a theory. On the other hand, if there is not an integer solution we must look to numerical results to help clarify the situation (at present). From previous numerical work, together with the associated eigenproblem approach of Miura, the number of solitons can be predicted. Unfortunately an oscillatory wave, which is associated with the continuous spectrum of eigenvalues, is also present and trails behind the solitons. In the case of the shelf problem, we expect the following picture. For any given d_0, (3) is satisfied if n is replaced by

$$p = -\tfrac{1}{2} + \tfrac{1}{2}(1 + 8 d_0^{-\frac{9}{4}})^{\frac{1}{2}},$$

and for certain d_0, p will be integer ($= n$). From Zabusky (1968) we have that the number of bound states of the related eigenvalue problem is N, where N is the largest integer satisfying $N \leqslant p+1$. When p is an integer, then $N = p+1$ and the number of solitons (i.e. bound states) is $p+1$, but one of these is of zero amplitude so that only p occur. However, when p is not an integer the latter soliton is no longer of zero amplitude. If we introduce

$$\Delta = 1 + p - N, \quad 0 \leqslant \Delta < 1,$$

then the amplitudes of the solitons as predicted by $(2a)$ are

$$\frac{24(\Delta + m)^2}{p(p+1)}, \quad m = 0, 1, \dots (N-1). \tag{7b}$$

Note that when $\Delta = 0$ this agrees with the formula $(7a)$. We can now predict that if d_0 is such that the solution of (3) lies between two integers, say N_0 and $N_0 + 1$, then eventually $N_0 + 1$ solitons will appear along the shelf, together with an oscillatory wave. If the solution is close to N_0, then the last (smallest) soliton will have a small (but non-zero) amplitude ($\propto \Delta^2$). Conversely, if the solution is close to $N_0 + 1$, then the final soliton will have an amplitude almost equal to $24/N_0(N_0+1)$ ($\Delta^2 \approx 1$). The purpose, then, of this paper is to confirm the above predictions for the variable-coefficient KV equation and to observe the oscillatory component of the final solution.

3. Numerical scheme

A difference equation that approximates the variable-coefficient KV equation (2) is

$$u_j^{i+1} = u_j^{i-1} - \tfrac{1}{3} d_i^{-\frac{7}{4}} (\Delta X/\Delta\xi)(u_{j+1}^i + u_j^i + u_{j-1}^i)(u_{j+1}^i - u_{j-1}^i) - d_i^{\frac{1}{2}} [\Delta X/(\Delta\xi)^3](u_{j+2}^i - 2u_{j+1}^i + 2u_{j-1}^i - u_{j-2}^i), \tag{8}$$

where $u_j^i = u(j\Delta\xi, i\Delta X)$ and $\Delta\xi$ and ΔX are the appropriate step lengths. The local depth is

$$d_i = d(i\Delta X). \tag{9}$$

The truncation error in (8) is proportional to $(\Delta X)^3$ and $\Delta\xi(\Delta X)^2$ (i.e. which ever is the larger). Since the scheme is centred in X, as well as in ξ, the initial step in X can be found by using a standard forward-integration procedure. Vleigenthart (1971) shows that (8) does not contain any numerical damping, but that the solution will grow indefinitely unless

$$\frac{\Delta X}{\Delta\xi}\left(d_i^{-\frac{7}{4}}|u| + \frac{4d_i^{\frac{1}{2}}}{(\Delta\xi)^2}\right) \leqslant 1. \tag{10}$$

It is also worth noting that the integration scheme conserves momentum and almost conserves energy (with error proportional to $(\Delta X)^2$).

The initial condition can be written as

$$u_j^0 = 12\,\mathrm{sech}^2(j\Delta\xi - c), \tag{11}$$

where c is a constant which prescribes the position of the peak of the solitary wave on the ξ mesh. The integration was performed with 500 steps in ξ and with c so chosen that there was the maximum undisturbed region ahead of the initial wave while ensuring that, behind this wave, the amplitude was as close to zero as desired. Although we are dealing with a wave motion it is best to use co-ordinates fixed in space. This is simply because the solitons produced move at different speeds corresponding to their amplitudes, and thus we could travel with only one soliton, at best.

The coarsest mesh that is worth considering for the satisfactory reproduction of the solitons is $\Delta\xi \approx 0{\cdot}1$. If we use this value in the stability criterion (10) and introduce the 'worst' values $|u| = 24, d_i = 1$, then we obtain $\Delta X \leqslant 1/4240$. Vliegenthart (1971) has discussed the merits of his criterion and found, from numerical trials, that if solitary waves are being propagated (10) could be exceeded by 50 % and still give stable solutions. With this in mind, and from trial runs by the author, it was decided to use

$$\Delta\xi = 0{\cdot}1, \quad \Delta X = 0{\cdot}00025. \tag{12}$$

At no time, using the values (12), were any instabilities encountered. It is evident that, owing to the very small steps in X, the integration must run for an appreciable time in order to see the development of the waves.

The shelf, $d(X)$, was described by a cosine function for all the results discussed in this paper. This particular form was chosen for its simplicity and continuous first derivatives when joined to a constant depth (see figure 1). The actual expression used was

$$d = \begin{cases} \frac{1}{2}[1 + d_0 + (1 - d_0)\cos(\pi j/40)] & (0 \leqslant j \leqslant 40), \\ d_0 \quad (j > 40). \end{cases} \tag{13}$$

Thus the region of changing depth was $0 \leqslant X \leqslant 0{\cdot}01$. Note that the detailed nature of the change in depth and its extent is irrelevant so far as the final soliton formation far along the shelf is concerned (see I). For the present integrations the region of change is of moderate length, being 40 steps in X.

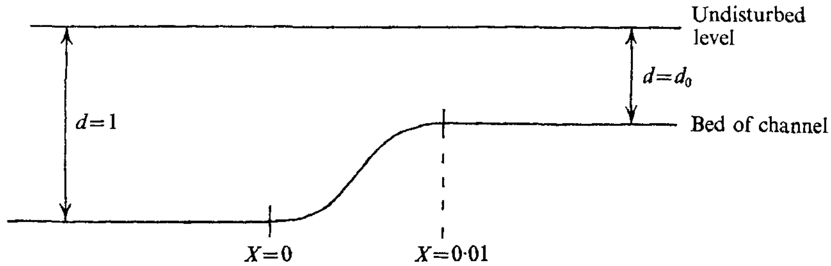

FIGURE 1. The geometry of the shelf.

4. Numerical integration results

For simplicity and ease of discussion we shall consider only the depths which correspond to the formation of two solitons ($n = 2, d_0 = 3^{-\frac{4}{9}} = 0{\cdot}614$), three solitons ($n = 3, d_0 = 6^{-\frac{4}{9}} = 0{\cdot}451$) and values of depth between these two. In fact, this is by far the most satisfactory region to choose since Madsen & Mei (1969) quote results for $d_0 = 0{\cdot}5$, and we can compute this solution and compare with their paper. The solutions at various values of X for a given shelf depth will be reproduced in a single figure. Thus it will be possible to see the development of the solitary wave as X changes, that is as the wave moves along the shelf.

It is worth making the point at this stage that, although it appears that the extent of the wave (approximately the period) more than covers the region of depth change, this is not the case. It must be remembered that in the original physical variables (5), the distance co-ordinate was stretched by $\epsilon^{-1}(\epsilon \to 0)$ but the local characteristic co-ordinate (ξ) was $O(1)$. Thus at each station X we may say that the wave profile takes a certain form as a function of ξ, i.e. for our KV equation X and ξ are independent variables. Hence the problem is equivalent to a wave which changes in *time* (X) in a medium whose inhomogeneity is described by $d(X)$.

The integration procedure outlined in §3 was performed on an IBM 360–67 machine in the Computing Department of the University of Newcastle. The solution was printed out as an array on the ξ mesh, after a specified number of integrations in steps of ΔX. The maximum runs employed gave solution values up to $X = 0{\cdot}5$. From the step length (12), we see that this corresponds to 2000 steps, which took, on average, about 1800 seconds of computing time. Even after such extended integrations, the only spurious oscillations ahead of the wave profiles were of amplitudes no larger than 0·0005 units, if they appeared at all. Such small values cannot be shown on the figures presented here. However, some oscillations (apart from those expected) did appear behind the profiles. These will be discussed later.

Figures 2 and 3 give the wave profiles for the solitary wave moving onto a shelf of depth given by expression (3). Figure 2 shows the breakup into exactly two solitons. The shelf depth is $d_0 = 0{\cdot}614$ ($n = 2$) and the soliton amplitudes (from (7*a*)) of 4 and 16 are indicated on the figure. Figure 3 shows the corresponding profiles for three solitons:

$$d_0 = 0{\cdot}451 \ (n = 3), \text{ soliton amplitudes} = 2, 8, 18.$$

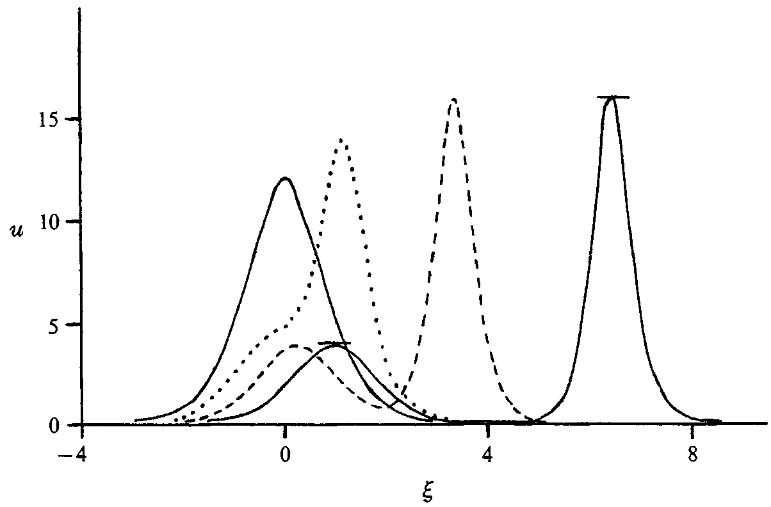

FIGURE 2. Two-soliton formation, $d_0 = 0{\cdot}614$. ——, initial solitary wave and solution at $X = 0{\cdot}5$; · · ·, solution at $X = 0{\cdot}075$; – – –, solution at $X = 0{\cdot}25$. Short horizontal lines indicate predicted maxima at 4 and 16.

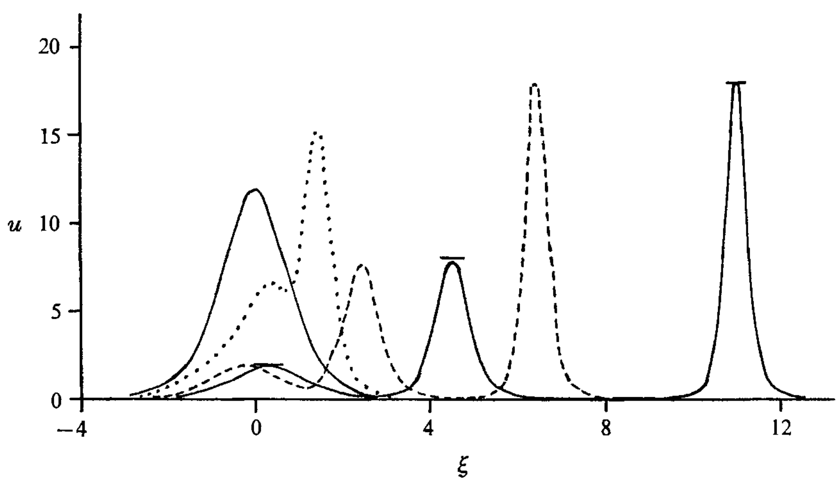

FIGURE 3. Three-soliton formation, $d_0 = 0{\cdot}451$. ——, initial solitary wave and solution at $X = 0{\cdot}45$; · · ·, solution at $X = 0{\cdot}05$; – – –, solution at $X = 0{\cdot}25$. Short horizontal lines indicate predicted maxima at 2, 8 and 18.

Figures 4 and 5 show the profile on the shelf for depths that lie between 0·614 and 0·451; $d_0 = 0{\cdot}5$ for the results plotted in figure 4 and $d_0 = 0{\cdot}55$ for figure 5. In these latter two figures the amplitudes predicted from (7*b*) are also given.

5. Discussion of the results

The formation of two and three solitons on the shelves of appropriate depth are confirmed in figures 2 and 3. It is also evident from the results that the solitary wave breaks up very slowly and consequently is virtually unchanged by

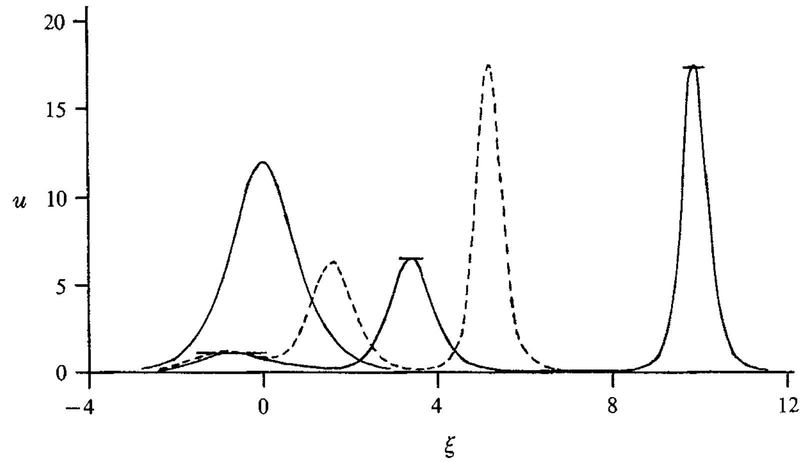

FIGURE 4. Intermediate depth, $d_0 = 0{\cdot}5$. ——, initial solitary wave and solution at $X = 0{\cdot}5$; – – –, solution at $X = 0{\cdot}25$. Short horizontal lines indicate predicted maxima.

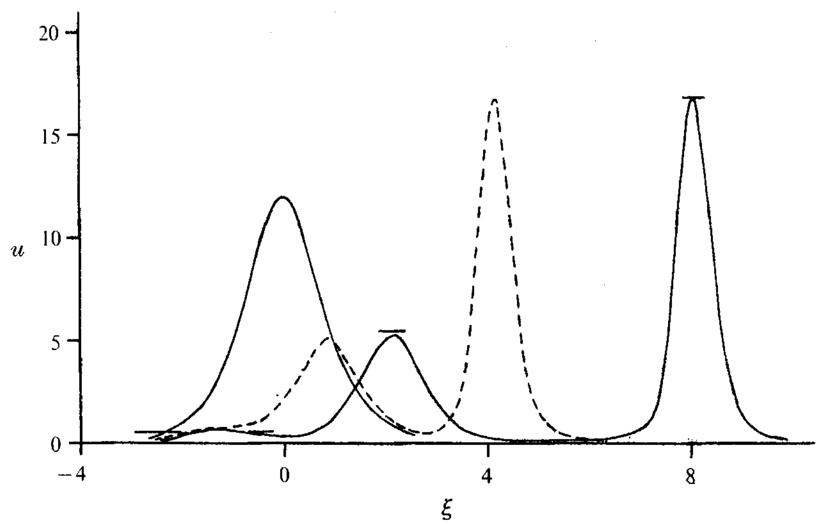

FIGURE 5. Intermediate depth, $d_0 = 0{\cdot}55$. ——, initial solitary wave and solution at $X = 0{\cdot}5$; – – –, solution at $X = 0{\cdot}25$. Short horizontal lines indicate predicted maxima.

the time it reaches the beginning of the shelf. (This tendency can also be observed in the work of Madsen & Mei (1969).) Thus we have the classical situation of a solitary wave which cannot propagate without deformation on the depth corresponding to the shelf. The wave then gradually develops to produce the appropriate number of solitons only. The soliton maxima predicted by the expression (7*a*) are also confirmed. That the amplitudes agree so well indicates that only the discrete eigenvalues are present (as predicted), the contribution from the continuous spectrum being zero in this case.

Turning to the other two integrations (figures 4 and 5), we must examine these solutions a little more carefully. At first sight, the solution for $d_0 = 0{\cdot}5$ is just three solitons, but we observe a number of differences. The most obvious

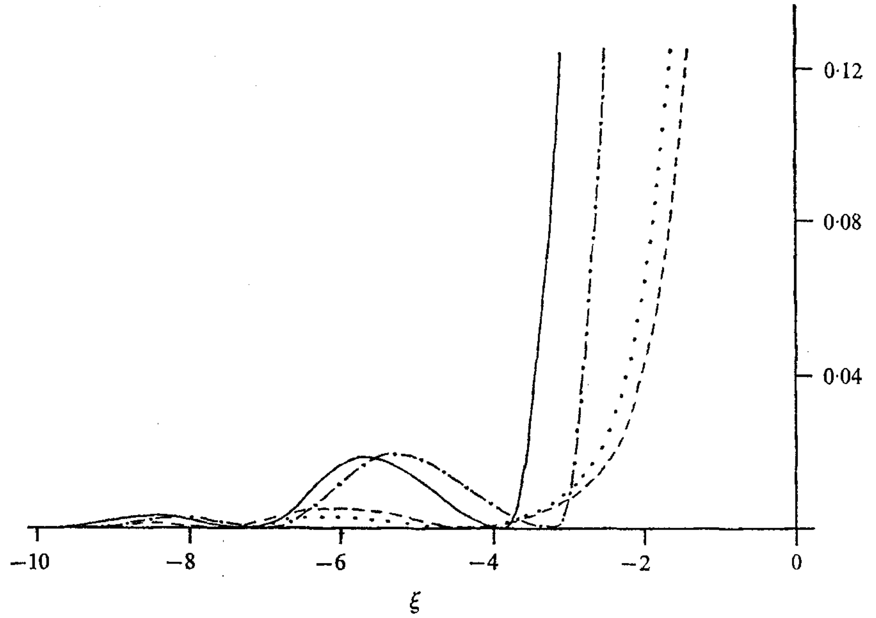

FIGURE 6. Oscillatory 'tails' of the four integrations all at $X = 0{\cdot}5$. The curves have the same origin (the peak of the initial solitary wave). Solutions: · · ·, $d_0 = 0{\cdot}451$; —·—, $d_0 = 0{\cdot}5$; ——, $d_0 = 0{\cdot}55$; – – –, $d_0 = 0{\cdot}614$.

is that only the first soliton (largest amplitude) is fully developed (i.e. essentially zero at both ends). The other two deviate considerably from a solitary wave profile, especially the last of the trio. Similar observations may be made for the depth $d_0 = 0{\cdot}55$ (figure 5), for which the final wave has an extremely small amplitude and is not yet fully developed.

Looking in more detail, we may compute the theoretical soliton amplitudes from the formula (7*b*). These are given in figures 4 and 5. We note that the agreement is very good indeed, confirming the validity of constant-coefficient KV theory in this variable-depth problem. Since the agreement is based on the assumption that both the continuous and discrete spectra contribute, an oscillatory wave is to be expected to appear to the left of the solitons. Thus it is instructive to examine the 'tails' of all the integrations (since these cannot be plotted on figures 2–5). In figure 6 the relevant portions of the four integrations at $X = 0{\cdot}5$ are reproduced on a greatly magnified scale. All the curves have the same origin, that is, the peak of the initial solitary wave. The oscillations behind the 'exact soliton' solutions, i.e. those for $d_0 = 0{\cdot}451$ and $0{\cdot}614$, are small ($< 0{\cdot}006$ units, or $0{\cdot}05\,\%$ of the initial profile). Since no oscillations can arise from the continuous spectrum in these two cases, these oscillations must be due to an accumulation of errors in the integration scheme. Such oscillations are well known and arise from the linear terms of the KV equation, which produce short-wave disturbances of the form $\exp i(\omega t - kx)$, propagating to the left. (These were encountered by Zabusky (1968) and Vliegenthart (1971), and arise from the short-wave behaviour discussed in detail by Benjamin, Bona & Mahony (1971). A smaller value of the mesh size ΔX would reduce these particular oscillations (ultimately to zero) but would, of course, increase the computing time (ultimately to infinity). A check to confirm this was made.) The occurrence of oscillations behind the solitons for the other two cases, $d_0 = 0{\cdot}5$ and $0{\cdot}55$, is to be expected even in the absence of numerical errors since they arise from the non-zero continuous spectrum in those cases. These oscillations were found to stay appreciably the same when ΔX was (slightly) altered.

From figure 5, apart from the obvious differences in amplitude between the integer and non-integer solutions, we observe that the oscillations for the two non-integer problems ($d_0 = 0{\cdot}5$ and $0{\cdot}55$) are very similar, which is perhaps rather surprising. Both these waves are of almost identical amplitude and shape, but slightly shifted in phase. Now Zabusky (1968) has shown that the momentum carried by these waves is proportional to $\Delta(1-\Delta)$, and has suggested that if the momenta (and energy, etc.) were equal for otherwise different solutions then the oscillatory parts would probably be very similar. In the present case, Δ has been found to be given by

$$\Delta = 0{\cdot}31, \quad \text{with} \quad \Delta(1-\Delta) = 0{\cdot}21 \quad \text{for} \quad d_0 = 0{\cdot}55,$$
$$\Delta = 0{\cdot}62, \quad \text{with} \quad \Delta(1-\Delta) = 0{\cdot}23 \quad \text{for} \quad d_0 = 0{\cdot}5,$$

values which agree to within 10 %. An attempt to check the momenta directly was made, but found impossible without extending the integrations, probably half as far again, to ensure that the solitons were sufficiently isolated from the oscillatory solution. Exactly the same difficulty was encountered by Zabusky (1968).

Finally, the results obtained here can be compared with those computed by Madsen & Mei (1969) for the shelf of depth 0·5. A similar comparison was made by Tappert & Zabusky (1971), using only theoretically predicted values. Direct agreement cannot be expected since the variable-coefficient KV equation was derived by introducing a single small parameter into the full equations and developing an asymptotic expansion. On the other hand, Madsen & Mei used essentially the full equations for integration purposes. However, the form of solution compares very well. For example, observe the great similarity between our solution for $X = 0{\cdot}25$ (figure 4) and the profile reproduced in figure 5(*c*) of Madsen & Mei's paper. It is encouraging to see that the variable-depth KV equation incorporates all the characteristics obtained from a more complete numerical study of the problem. To check more than just qualitative agreement we can compare the amplitudes of the first three peaks. From figure 4, and Madsen & Mei (1969), we have the results shown in table 1, in which Madsen & Mei's results have been normalized to a solitary wave amplitude of 12. Note that the amplitudes quoted from figure 4 and expression (7*b*) have been increased by the factor $d_0^{-\frac{1}{4}}$ (see §2). It is clear that the agreement is fairly good but the two smaller peaks are slightly in error (in particular the smallest one; this is probably due to this soliton not being fully developed). With the differences outlined above, together with no detailed knowledge of the initial profile used by Madsen & Mei, the results in table 1 might well be regarded as satisfactory.

In conclusion, this paper has examined some numerical solutions of a variable-coefficient Korteweg–de Vries equation. The particular equation discussed was

		Initial wave	Peak 1	Peak 2	Peak 3
Present results	prediction (7*b*)	12	20·5	7·88	1·16
	numerical	12	20·8	7·8	1·3
Madsen & Mei		12	20	9	2

TABLE 1

previously derived by the author in a study on solitary waves moving onto a shelf (I), and the results have been presented from this point of view. Solutions for various shelf depths have been obtained. In particular, the two- and three-soliton formation has been confirmed and the wave profiles corresponding to depths intermediate between these two cases have been analysed in some detail. A fair agreement has been obtained with the full integrations of Madsen & Mei (1969), who in turn obtained some agreement with experimental data.

One point not studied in detail in this paper is the effect, if any, of changing the form of the depth variation and altering X_0, for a *given* shelf depth. Such a numerical study would, hopefully, validate the theoretical results used here and give some insight into the corresponding conservation laws for the equation. This comprehensive examination was thought to be beyond the direct aim of this paper, but the author intends to begin such a protracted study now that the relevance of the equation appears justified.

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